Tremonia Fechterkurier

Concerning the Dynamics of Swords

There are two major models that specify the point at which a sword should ideally hit its target. One model focusses on the sword's vibration, particularly the nodes of the fundamental flexural vibration; the node which is closer to the sword's point is often called *centre of percussion.* [1, 2] The other model, which is the subject of this work, considers translation and rotation as the components of a rigid body's motion [3] upon impact. Several comprehensive articles on the rigid-body dynamics of swords exist, such as by Turner [4], Denny [5] and Le Chevalier [6]; those articles are highly recommended for a further reading on the consequences and possible applications of a sword's analysis on the basis of rigid-body dynamics. The physical principles have been known since the 17th century. [7] Rigid-body dynamics were already applied in some historical sources on fencing [8, 9] to describe the point where a cutting weapon should ideally hit its target.

This article is meant as a brief introduction of the subject for those who are not yet aware of the physics of fencing. Additionally, this article emphasises the importance of the moment of inertia for the behaviour of a sword, and encourages sword researchers and makers to determine and include this fundamental measure from which other parameters can be calculated.

1 Motion of a Sword

At any given time, a sword's motion can be described as rotation about an axis, translation along a trajectory and as vibration. As we focus on sword hits with the edge, i. e. sword use with no significant bending of the blade, we neglect vibration and consider the sword an ideally rigid body. Every body has a resistance against a change of its translation, i. e. mass, and against a change of its rotation, i. e. moment of inertia. Each mass point m_i with the distance r_i from the axis has a moment of inertia $I_i = m_i r_i^2$. The total moment of inertia I is the sum of the moments of inertia of its parts:

$$I = \sum_i m_i r_i^2 \tag{1}$$

The moment of inertia not only depends on the mass distribution of the body but also on the body's position relative to the axis about which it rotates. The axis about which the moment of inertia is smallest passes through the centre of mass and is called the third principal axis. The perpendicular axes which intersect the third principal axis in the centre of mass are the first and the second principal axis. The first principal axis is the principal axis with the largest moment of inertia. It is also the first principal axis which is most relevant for the sword's cutting motion. While the sword does not necessarily rotate about the first principal axis during a strike, it is commonly an axis which is parallel to the first principal axis. For an axis parallel to an axis through the centre of mass M for which the moment of inertia I_M is known, the moment of inertia I_x can be calculated according to the parallel-axis theorem with the distance r_x between the axis of rotation and the parallel axis through the centre of mass:

$$I_x = I_M + m r_x^2 \tag{2}$$

Note that the moment of inertia I_x is larger when the axis of rotation is situated further away from the centre of mass M. As the moment of inertia describes the resistance to a change of rotation, this means that weapons like axes or maces with more distance between the axis of rotation and the centre of mass are in general more difficult for the fencer to manoeuvre but also more difficult for the opponent's armour to stop. The larger distance between axis of rotation and centre of mass and the thus larger moment of inertia is what makes the difference between a *Mordhau* and an *Oberhau*.

1.1 The Sword as a Physical Pendulum

A point mass which can rotate about an axis in a homogeneous gravitational field is called a mathematical pendulum. In contrast, an extended rigid object of arbitrary shape which can rotate about an axis in a homogeneous gravitational field is called a physical pendulum. The angular eigenfrequency ω_0 of a mathematical pendulum is defined by its length l and the gravitational acceleration g. The length of a mathematical pendulum which oscillates at the same angular eigenfrequency ω_0 as a physical pendulum is called *radius of oscillation* or *equivalent pendulum length* l_r. The point which is situated l_r below the pivot P when the pendulum is at rest is called the *centre of oscillation*. In a mathematical pendulum, the centre of mass and the centre of oscillation are the same point, while in a physical pendulum, these points are generally in different positions.

Radius of Oscillation

To calculate the radius of oscillation and thus the position of the centre of oscillation, we start with the conservation of energy. In the frictionless case, the sum of rotational

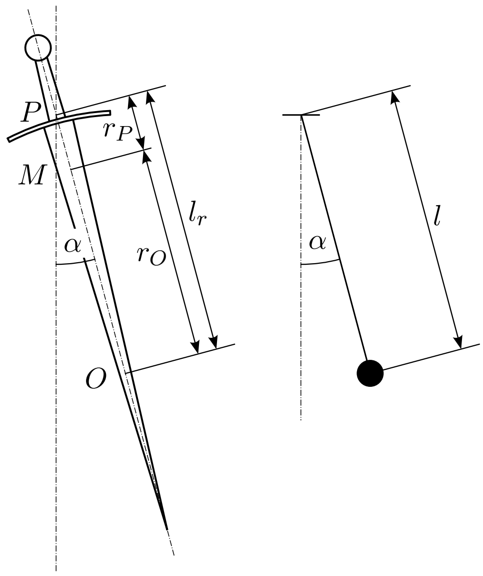

Figure 1: The sword as a physical pendulum and a mathematical pendulum with the same angular eigenfrequency ω_0

energy E_{rot} and potential energy E_{pot} remains constant:

$$E_{\text{total}} = E_{\text{rot}} + E_{\text{pot}} \tag{3}$$

The rotational energy is $E_{\text{rot}} = I_P\dot{\alpha}^2/2$ with the moment of inertia I_P about the pivot P and the angular velocity $\dot{\alpha} = d\alpha/dt$. The potential energy of a mass m in the gravitational field with a gravitational acceleration g and an angular displacement α is $E_{\text{pot}} = mgr_P(1-\cos\alpha)$, which is approximated as $E_{\text{pot}} \approx mgr_P\alpha^2/2$ for small angular displacements $\alpha \ll \pi/2$. With Eq. (3), this yields:

$$E_{\text{total}} = \frac{I_P}{2}\dot{\alpha}^2 + \frac{mgr_P}{2}\alpha^2 \tag{4}$$

Deriving with respect to time t results in the simple oscillation equation:

$$0 = I_P\ddot{\alpha} + mgr_P\alpha \tag{5}$$

Thus, the angular displacement is $\alpha = \alpha_0 \sin(\omega_0 t + \varphi)$ with the amplitude α_0, the phase angle φ and the angular eigenfrequency

$$\omega_0 = \sqrt{\frac{mgr_P}{I_P}} \tag{6}$$

A mathematical pendulum with the same eigenfrequency ω_0 has the length $l = g/\omega_0^2$. Thus, using Eq. (6), the radius of oscillation l_r of a physical pendulum, which is the distance between the pivot P and the centre of oscillation O, is

$$l_r = \frac{I_P}{mr_P} \tag{7}$$

It can be proven [3, 10] that the relation of the pivot P and the centre of oscillation O is symmetric, i. e. if O is the centre of oscillation for the pivot P, then P is the centre of oscillation for an oscillation about the pivot O. Thus, the oscillations about P and O have the same radius of oscillation l_r and the same angular eigenfrequency ω_0.

A mathematical pendulum with the radius of oscillation l_r of a physical pendulum has the same eigenfrequency ω_0 and – as the eigenfrequency of a mathematical pendulum is independent of its mass – can have either the same mass or the same moment of inertia as a physical pendulum, but usually not both of them. The motion of a physical pendulum is defined by three parameters, e. g. mass m, distance r_P between the pivot and the centre of mass, and the distance l_r between the pivot and the centre of oscillation. In a mathematical pendulum, r_P, l_r and the radius of gyration R_g degenerate into the pendulum length l. As a mathematical pendulum is defined by two parameters (length and mass), it cannot model every aspect of a physical pendulum defined by three parameters. Therefore, Turner [4] and Le Chevalier [6] suggest a two-mass model to describe the dynamic aspects of a sword.

Side Note: Radius of Gyration

The radius of oscillation l_r is not to be confused with the *radius of gyration* R_g. The radius of gyration R_g is defined as the distance from the pivot P, where all the mass m of a physical pendulum would be situated to result in a body with the same mass m and the same moment of inertia I_P as the original physical pendulum, that is

$$R_g = \sqrt{\frac{I_P}{m}} \tag{8}$$

and thus $R_g^2 = r_P l_r$. The radius of gyration R_g characterises the *mass distribution* about a given pivot P. Substituting I_P with mR_g^2 in Eq. (6), we find

$$\omega_0^2 = \frac{mgr_P}{I_P} = \frac{mgr_P}{mR_g^2} = \frac{gr_P}{R_g^2} \tag{9}$$

Thus, the eigenfrequency of a physical pendulum depends not on the mass itself, but on the mass distribution characterised by the radius of gyration R_g. Two homogeneous physical pendulums of different density but identical geometric dimensions and equivalent axes have the same eigenfrequency.

1.2 The Sword and the Impact

A force F that is exerted on a rigid body in a point different from the centre of mass M can be substituted with a force F' of the same value in the centre of mass M and a moment of force τ' about the centre of mass M. If a sword is considered a physical pendulum with the pivot P, the centre of oscillation O, the radius of oscillation l_r and mass m, then the acceleration $a_{F'}$ through the force F' and the acceleration $a_{\tau'}$ through the moment of force τ' cancel out each other in the pivot P, when the force F is exerted in the centre of oscillation O.

The acceleration $a_{F'}$ through the force F' is $a_{F'} = F'/m$ in P just as in M. The angular acceleration $\ddot{\alpha}$ about the centre of mass M through the moment of force $\tau' = Fr_O$ is $\ddot{\alpha} = \tau'/I_M = Fr_O/I_M$. The linear acceleration in the pivot P through $\ddot{\alpha}$ is $a_{\tau'} = \ddot{\alpha} r_P = Fr_P r_O/I_M$. The moment of inertia I_M about the centre of mass M is, according to the parallel-axis theorem (2), expressed as $I_M = I_P - mr_P^2$. Thus,

$$a_{\tau'} = \frac{Fr_P r_O}{I_P - mr_P^2} \tag{10}$$

According to Eq. (7), we can substitute I_P with $mr_P l_r$ and get the result

$$a_{\tau'} = \frac{Fr_P(l_r - r_P)}{mr_P l_r - mr_P^2} = \frac{F}{m} = a_{F'} \tag{11}$$

This means that a sword which hits in the centre of oscillation O will not impart any acceleration (but a moment of force) in the corresponding pivot P upon impact.

2 Measurement

As shown above, a rigid body's response to external forces can be calculated if the mass m, the centre of mass M and the moment of inertia I are known. Without the moment of inertia, a body's response to non-central forces cannot be calculated. From Eqs. (2) and (7), we know that the moment of inertia can be deduced from the radius of oscillation and vice versa.

Turner [4] and Le Chevalier [6] suggest the so-called "waggle test" as a method to determine the radius of oscillation l_r. For this procedure, the sword is loosely held at a pivot point and carefully shaken in the directions of the edges. The point of the sword that remains without acceleration is the corresponding centre of oscillation. The advantage of this experiment is that it does not require additional equipment. However, despite the authors' thorough explanations on how to properly conduct the measurement, it should be kept in mind that the accuracy and precision of the corresponding points' positions depend on the experimenter's ability to manually shake a sword with no radial component while spotting the point that moves least on a shaking sword.

The moment of inertia is relevant not only for the dynamics of swords, but also for the automotive industry, the assessment of golf clubs and various other applications. While the moment of inertia can be easily calculated from exact density values using CAD, the precision of the input values may be insufficient to produce an accurate value for the moment of inertia. Thus, it is convenient to measure the moment of inertia. Usually, the object is therefore oscillated using a trifilar torsion pendulum or a spiral spring pendulum. From the period of oscillation, the object's moment of inertia can be calculated. [10]

3 Conclusion

Mass and the position of the centre of mass together with the geometric dimensions are not sufficient to define a rigid body's dynamic properties. Researchers and makers who want to characterise blunt or cutting weapons should also determine the weapon's moment of inertia for at least the first principal axis or a parallel axis in order to provide meaningful numbers that define the weapon's motion. The moment of inertia can be measured using a rotating pendulum. [10]

Alternatively, the moment of inertia can be deduced from the radius of oscillation (and vice versa), which can be measured with the "waggle test" with no additional equipment. However, the accuracy of this method heavily relies on the experimenter's skill and should thus not be preferred.

In the context of sword characterisation, the vibrational node near the point as well as the centre of oscillation are mistakably both called *centre of percussion*. To avoid confusion, it is suggested to use non-ambiguous terms such as *vibrational node* or *centre of oscillation* with respect to a given pivot, respectively.

symbol	explanation
a	acceleration
E_{pot}	potential energy
E_{rot}	rotational energy
F, F'	force
g	gravitational acceleration
I_x	moment of inertia with respect to axis x
l	length of a mathematical pendulum
l_r	radius of oscillation; effective length [4]; dynamic length [6]
m	mass
M	centre of mass; centre of gravity; point of balance
O	centre of oscillation; centre of percussion
P	pivot point of a pendulum
R_g	radius of gyration
r_x	distance between x and the centre of mass
t	time
α	angular displacement
$\dot{\alpha}$	angular velocity
$\ddot{\alpha}$	angular acceleration
α_0	amplitude; maximum angular displacement
π	ratio of a circle's circumference to its diameter; $3.14159\ldots$
τ, τ'	moment of force; torque
φ	phase angle
ω_0	angular eigenfrequency

Table 1: Notation

References

[1] T. Laible. *Das Schwert – Mythos und Wirklichkeit.* 2nd ed. Wieland Verlag, 2008.

[2] H. Schmidt. *Vibration of the blade and how to use it.* Hroarr. 2014. URL: `http://www.hroarr.com/vibration-of-the-blade-and-how-to-use-it/`.

[3] H. Vogel. *Gerthsen Physik.* 20th ed. Springer, 1999.

[4] G. L. Turner. *Dynamics of Hand-Held Impact Weapons.* Association for Renaissance Martial Arts. 2002. URL: `http://armor.typepad.com/bastardsword/sword_dynamics.pdf`.

[5] M. Denny. "Swordplay: an exercise in rotational dynamics". In: *European Journal of Physics* 27.4 (July 2006), pp. 943–950. DOI: `10.1088/0143-0807/27/4/025`.

[6] V. Le Chevalier. *A dynamic method for weighing swords.* Ensis Sub Caelo. 2010. URL: `http://www.subcaelo.net/ensis/weighing/weighing.pdf`.

[7] C. Huygens. *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricæ.* F. Muguet, 1673.

[8] A. Fehn. *Die Fechtkunst mit Stoß- und Hiebwaffen.* Carl Rümpler, 1851.

[9] R. F. Burton. *The Book of the Sword.* Chatto & Windus, 1884.

[10] T. Dorfmüller, W. T. Hering and K. Stierstadt. *Bergmann/Schaefer Lehrbuch der Experimentalphysik.* 11th ed. Vol. 1. Walter de Gruyter, 1998.